ALL IN ONE Mega File
MTH101 Midterm PAPERS,
MCQz & subjective
Created BY Farhan & Ali
BS (cs) 3rd sem
Hackers Group
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Remember us in your prayers

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Question No: 24 (Marks: 10)

Evaluate the following limit Q # 1 Whether the given lines are parallel, perpendicular or none of these?

$$\frac{1}{2}(y-1) = x-3 \;\; and \;\; 8-2y = x+7$$

Solution:

$$L_1 : \frac{1}{2}(y-1) = x-3$$
$$L_2 : \quad 8-2y = x+7$$

First we have to calculate the slope.

$$L_1 : \frac{1}{2}(y-1) = x-3$$

$$y-1 = 2(x-3)$$

$$y = 2x-6+1$$

$$y = 2x-5$$

Comparing it with equation of line y = m x + c

Slope of $L_1 = m_1 = 2$

$$L_2 : 8-2y = x+7$$

$$-2y = x+7-8$$

$$-2y = x-1$$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

Comparing it with equation of line y = m x + c

Slope of $L_2 = m_2 = -1/2$

Hence the given two lines are perpendicular because $m_1 m_2 = (2)$

$(-\frac{1}{2}) = -1$

Q # 2

Let

$$f(x) = \frac{x}{x+3} \quad \text{and} \quad g(x) = x^2$$

Find whether $(f \circ g)(x)$ and $(g \circ f)(x)$ are equal or not?

Solution:

Here

$f(x)=\dfrac{x}{x+3}$ and $g(x)=x^2$

$$(f\circ g)(x)=f\,(\,g\,(x))$$
$$(f\circ g)(x)=f\,(\,x^2\,)$$
$$(f\circ g)(x)=\frac{x^2}{x^2+3}$$

Also

$$(g\circ f)(x)=g\,(\,f\,(\,x\,))$$
$$(g\circ f)(x)=g\,(\frac{x}{x+3})$$
$$(g\circ f)(x)=\frac{x^2}{(\,x+3\,)^2}$$
$$(g\circ f)(x)=\frac{x^2}{x^{\,2}+6x+9}$$

Hence $(f\circ g)(x)\neq(g\circ f)(x)$

Q # 3 Determine whether the equation represents a circle, if the equation represents a circle, find the center and radius?

$x^2 + 2x + y^2 - 4y = 5$

<u>Solution:</u>

First, group the x-terms, group the y-terms, and take the constant to the right side:

$$(x^2 + 2x) + (y^2 - 4y) = 5$$

$$x^2+2x +1+y^2-4y +4=5+1+4$$

Then write the left side as squares and add up the right side and you get

$$(x+1)^2+(y-2)^2=10,$$

and now you can find the center and radius and graph it. Here the center would be (-1,2) and the radius would be $\sqrt{10}$. So to graph it, all you need to do is find the point (-1,2) and then plot the points $\sqrt{10}$ up, $\sqrt{10}$ down, $\sqrt{10}$ to the right, and $\sqrt{10}$ to the left of it, and draw the circle through them.

Q # 4 Solve the inequality

$$-6 \le \frac{4-2x}{3} < 1$$

<u>Solution:</u>

$$-6 \le \frac{4-2x}{3} < 1$$

multiply by 3, *we get*

$$-18 \le 4 - 2x < 3$$

subtracting by 4

$$-22 \le -2x < -1$$

divide by -2

$$11 \ge x > \frac{1}{2}$$

or $\frac{1}{2} < x \le 11$

Hence the required solution is

$$\left(\frac{1}{2}, 11\right]$$

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Slope of L_2 = m_2 = - 1 / 2

Hence the given two lines are perpendicular because $m_1 m_2$ = (2) $(-\frac{1}{2})$ = -1

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<u>Solution:</u>

Here

$$f(x) = \frac{x}{x+3} \quad \text{and} \quad g(x) = x^2$$

$$(f \circ g)(x) = f\,(\,g\,(x)\,)$$
$$(f \circ g)(x) = f\,(\,x^2\,)$$
$$(f \circ g)(x) = \frac{x^2}{x^2+3}$$

Also

$$(g \circ f)(x) = g(f(x))$$
$$(g \circ f)(x) = g\left(\frac{x}{x+3}\right)$$
$$(g \circ f)(x) = \frac{x^2}{(x+3)^2}$$
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$Hence\ the\ required\ solution\ is$

$$\left(\frac{1}{2}, 11\right]$$

$$\lim_{y\to -2} g(y)\ \ where,\ g(y) = \begin{cases} y^2+5 & if\ \ y<-2 \\ 3-3y & if\ \ y \ge -2 \end{cases}$$

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Question No.1

Use implicit differentiation to find dy/dx if

$$x^2 = \frac{x+y}{x-y}$$

Solution:

$$x^2 = \frac{x+y}{x-y}$$

$$\frac{d(x^2)}{dx} = \frac{(x-y)\frac{d(x+y)}{dx}) - (x+y)(\frac{d(x-y)}{dx})}{(x-y)^2}$$

$$2x = \frac{(x-y)(1+\frac{dy}{dx}) - (x+y)(1-\frac{dy}{dx})}{(x-y)^2}$$

$$2x = \frac{x-y+x\frac{dy}{dx} - y\frac{dy}{dx} - x - y + x\frac{dy}{dx} + y\frac{dy}{dx}}{(x-y)^2}$$

$$2x = \frac{-2y + 2x\frac{dy}{dx}}{(x-y)^2}$$

$$2x(x-y)^2 = -2y + 2x\frac{dy}{dx}$$

$$x(x-y)^2 = -y + x\frac{dy}{dx}$$

$$x(x-y)^2 + y = x\frac{dy}{dx}$$

$$\frac{x(x-y)^2 + y}{x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x(x-y)^2 + y}{x}$$

Some other method can be used to solve this.

Question No. 2.

Find the slope of the tangent line to the given curve at the specified point

$$2(x^2+y^2)^2 = 25(x^2-y^2); \qquad (3,1)$$

Solution:

$$2(x^2+y^2)^2 = 25(x^2-y^2)$$

differentiating with respect to 'x'

$$2\times 2(x^2+y^2)\frac{d(x^2+y^2)}{dx} = 25\frac{d(x^2-y^2)}{dx}$$

$$4(x^2+y^2)(2x+2y\frac{dy}{dx}) = 25(2x-2y\frac{dy}{dx})$$

$$4(x^2+y^2).2(x+y\frac{dy}{dx}) = 2(25x-25y\frac{dy}{dx})$$

$$8(x^2+y^2)(x+y\frac{dy}{dx}) = 2(25x-25y\frac{dy}{dx})$$

$$4(x^2+y^2)(x+y\frac{dy}{dx}) = (25x-25y\frac{dy}{dx})$$

$$4x(x^2+y^2)+4y(x^2+y^2)\frac{dy}{dx} = 25x-25y\frac{dy}{dx}$$

$$4y(x^2+y^2)\frac{dy}{dx}+25y\frac{dy}{dx} = -4x(x^2+y^2)+25x$$

$$4y(x^2+y^2)\frac{dy}{dx}+25y\frac{dy}{dx} = -x(4x^2+4y^2)+25x$$

$$y(4x^2+4y^2)\frac{dy}{dx}+25y\frac{dy}{dx} = x(-4x^2-4y^2+25)$$

$$y(4x^2+4y^2+25)\frac{dy}{dx} = x(-4x^2-4y^2+25)$$

$$\frac{dy}{dx} = \frac{x(-4x^2-4y^2+25)}{y(4x^2+4y^2+25)}$$

At point (3,1)

$$\left.\frac{dy}{dx}\right|_{(3,1)} = \frac{x(-4x^2 - 4y^2 + 25)}{y(4x^2 + 4y^2 + 25)}$$

$$\left.\frac{dy}{dx}\right|_{(3,1)} = \frac{3(-4\times 3^2 - 4\times 1^2 + 25)}{1(4\times 3^2 + 4\times 1^2 + 25)}$$

$$= \frac{3(-36 - 4 + 25)}{36 + 4 + 25}$$

$$\left.\frac{dy}{dx}\right|_{(3,1)} = \frac{-3\times 15}{40 + 25}$$

$$= \frac{-45}{65}$$

$$= \frac{-9}{13}$$

Question No. 3

A 10 –f t ladder is leaning against a wall. If the top of the ladder slips down the wall at the rate of 2- ft/sec, how fast will the foot be moving away from the wall when the top is 6 ft above the ground?

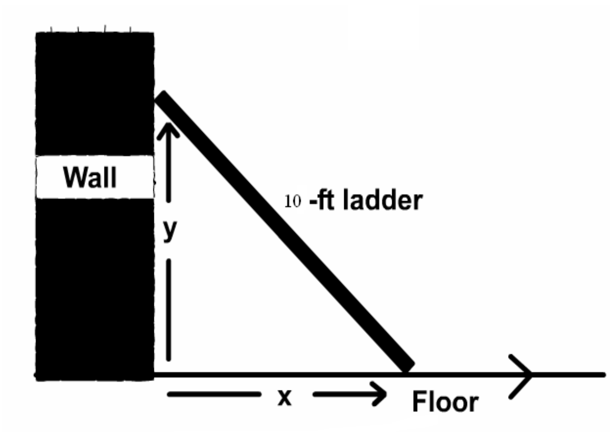
Solution:

Let

x = Distance in feet between wall and foot of the ladder

y = Distance in feet between floor and top of the ladder

t = number of seconds after the ladder starts to slip.



$$\frac{dy}{dt} = -2\ ft/\sec$$

It is negaive because y is decreasing as time increases, since the ladder is slipping down the wall.

$$\frac{dx}{dt} = ?$$

Using Pythagoras Theorem,

$$x^2 + y^2 = 10^2$$

differentiating w.r.t 't'

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

when $y = 6\,ft$

$$\frac{dx}{dt} = -\frac{6}{x}\frac{dy}{dt} \quad ....(i)$$

When the top is 6 ft above the ground

$$x^2 + 6^2 = 10^2$$

$$x^2 = 100 - 36$$

$$x^2 = 64$$

$$x = 8$$

put in (i) we get

$$\frac{dx}{dt} = -\frac{6}{8}\frac{dy}{dt}$$

$$\frac{dx}{dt} = -\frac{6}{8}(-2)$$

$$= \frac{12}{8}$$

$$= \frac{3}{2}\,ft/\sec$$

Question No.1 :

Find the relative extreme values of the function

$$f(x) = a \sin x + b \cos x$$

Solution:

Since,

$$f(x) = a \sin x + b \cos x$$

$f'(x) = a\cos x - b\sin x$...................$(i)$

put $f'(x) = 0 \Rightarrow a\cos x - b\sin x = 0$

$\Rightarrow a\cos x = b\sin x$

$\Rightarrow \frac{a}{b} = \tan x$

$\Rightarrow \tan x = \frac{a}{b} \Rightarrow x = \tan^{-1}(\frac{a}{b})$

By Pythagorus theorem

$\Rightarrow \sin x = \frac{a}{\pm\sqrt{a^2+b^2}} \quad , \quad \cos x = \frac{b}{\pm\sqrt{a^2+b^2}}$

Now, taking second derivative

$f''(x) = -a\sin x - b\cos x$.....................$(ii)$

put $\quad \sin x = \frac{a}{\sqrt{a^2+b^2}}, \quad \cos x = \frac{b}{\sqrt{a^2+b^2}}$ *in* (ii)

$f''(x) = -a\frac{a}{\sqrt{a^2+b^2}} - b\frac{b}{\sqrt{a^2+b^2}}$

$f''(x) = -(\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}) = -\sqrt{a^2+b^2} < 0$

So $f(x)$ *has relative maxima at* $\sin x = \frac{a}{\sqrt{a^2+b^2}}$ *and* $\cos x = \frac{b}{\sqrt{a^2+b^2}}$

and since both sin and cos are positve in first quadrant so we say f has relative maxima at

$x = \frac{\pi}{2} - \tan^{-1}(\frac{a}{b})$

put $\quad \sin x = \frac{-a}{\sqrt{a^2+b^2}}, \quad \cos x = \frac{-b}{\sqrt{a^2+b^2}}$ *in* (ii)

$f''(x) = a\frac{a}{\sqrt{a^2+b^2}} + b\frac{b}{\sqrt{a^2+b^2}}$

$f''(x) = (\frac{a^2}{\sqrt{a^2+b^2}} + \frac{b^2}{\sqrt{a^2+b^2}}) = \sqrt{a^2+b^2} > 0$

So $f(x)$ *has relative minima at* $\sin x = \frac{-a}{\sqrt{a^2+b^2}}$ *and* $\cos x = \frac{-b}{\sqrt{a^2+b^2}}$

and since both sin and cos are positve in third quadrant so we say f has relative minima at

$x = \frac{3\pi}{2} - \tan^{-1}(\frac{a}{b})$

Question No.2:

Let

$$f(x)=\begin{cases} x^2, & if\ \ x\le 1 \\ x\ , & if\ \ x>1 \end{cases}$$

Does Mean Value Theorem hold for f on $\left[\frac{1}{2},2\right]$?

Solution:

To check whether f satisfies M.V.T

I. Clearly f is continuous on $\left[\frac{1}{2},2\right]$.

II. To check f is differentiable on $\left]\frac{1}{2},2\right[$

$We\,check\,whether\ f'(1)\,exists.$

$$L.H.Limt\ f'(1)=\lim_{x\to 1-0}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1-0}\frac{x^2-1}{x-1}$$

$$=\lim_{x\to 1-0}x+1=2$$

$$R.H.Limt f'(1)=\lim_{x\to 1+0}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1-0}\frac{x-1}{x-1}=1$$

$Thus\ L.H.Limt\ f'(1)\ne R.H.Limt f'(1).$

$Therefore,\ f'(1)\,does\,not\,exist\,and\,the\,M.V.T.\,does\,not\,hold\,on\left[\frac{1}{2},2\right].$

Question No.3 :

Integrate the following

$(i)\quad \dfrac{xa^{x^2}}{x^2} \qquad (ii)\quad \dfrac{\ln x}{x}$

Solution:

$$(i)\ \int \frac{xa^{x^2}}{x^2}dx$$

$$put\ x^2 = t \Rightarrow 2xdx = dt \Rightarrow xdx = \frac{dt}{2}$$

$$\int \frac{a^t dt}{2t} = \frac{1}{2}\int \frac{a^t}{t}dt = \frac{1}{2}\int [\frac{1}{t}(1 + t\ln a + \frac{t^2}{2!}(\ln a)^2 + \frac{t^3}{3!}(\ln a)^3 + ...)]$$

$$(\because a^x = 1 + x\ln a + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + ...)$$

$$= \frac{1}{2}\int [\frac{1}{t} + \ln a + \frac{t}{2!}(\ln a)^2 + \frac{t^2}{3!}(\ln a)^3 + ...)]$$

$$= \frac{1}{2}(\ln t + t\ln a + \frac{t^2}{4}(\ln a)^2 + \frac{t^3}{18}(\ln a)^3 + ...)$$

$$= \frac{1}{2}(\ln x^2 + x^2\ln a + \frac{(x^2)^2}{4}(\ln a)^2 + \frac{(x^2)^3}{18}(\ln a)^3 + ...)$$

$$(ii)\ \int \frac{\ln x}{x}dx$$

$$put\ lnx = t \Rightarrow \frac{1}{x}dx = dt$$

$$\int tdt = \frac{t^2}{2} + c$$

By back substitution

$$\int tdt = \frac{(\ln x)^2}{2} + c$$

Question 1

Marks 8

Find area of region enclosed by given curves

$y = 3x - 4x^2 + x^3 \quad , \quad y = 0, x = 0, x = 3.$

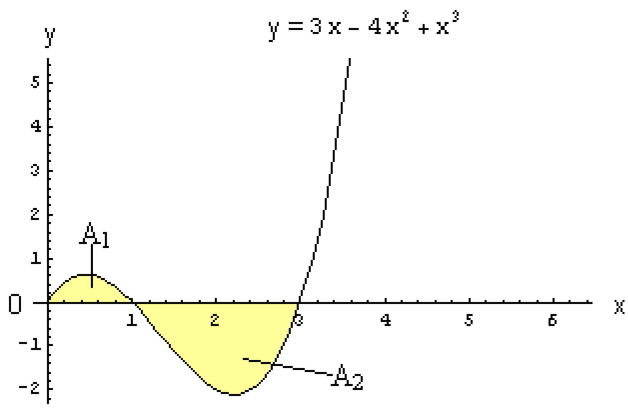
Solution:

$y = x^3 - 4x^2 + 3x \quad , \quad y = 0, x = 0, x = 3.$

y = 0 is the x-axis, x = 0 is y-axis and $x = 3$ is a line parallel to y-axis crossing x-axis at point 3.

So we have to find out the area bounded by the curve $y=x^3-4x^2+3x$ and the x-axis over the interval [0,3]

Graph of $y = 3x - 4x^2 + x^3$ is below



We have to find the area of shaded region. It is clear from the graph that total area, A, under the curve in interval [0,3] is divided into two regions A_1 and A_2. In [0,1], A_1 is above x-axis and in [1,3], A_2 is below x-axis. So

$$A = A_1 - A_2 = \int_0^1 \left(x^3 - 4x^2 + 3x\right)dx \; - \; \int_1^3 \left(x^3 - 4x^2 + 3x\right)dx$$

NOTE: Since $A_2 = \int_1^3 \left(x^3 - 4x^2 + 3x\right)dx$ is below x-axis , so we will get negative value of integral. That's why we subtract A_2 from A_1.

Even if we don't have the graph, we can come to this conclusion by first finding the zero of a function that lie between given interval [0,3].

Zero of a function is a value of x which makes a function f(x) equal to zero.

$$x^3 - 4x^2 + 3x = 0$$
$$x(x^2 - 4x + 3) = 0$$
$$x(x^2 - 3x - x + 3) = 0$$

$$x\{x(x-3) - 1(x-3)\} = 0$$
$$x(x-3)(x-1) = 0$$
$$x = 0,1,3$$

So zero of a function $\left(x^3 - 4x^2 + 3x\right)$ is 0, 1 and 3

Now 1 lie between our given interval [0, 3], which means that total area is divided into two regions. One in interval [0,1] and other in [1,3]

So

$$A = A_1 - A_2 = \int_0^1 \left(x^3 - 4x^2 + 3x\right)dx \ - \ \int_1^3 \left(x^3 - 4x^2 + 3x\right)dx$$

$$= \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right)\Bigg|_0^1 - \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{3x^2}{2}\right)\Bigg|_1^3$$

$$= \frac{1^4}{4} - \frac{4(1)^3}{3} + \frac{3(1)^2}{2} - \left(\frac{0^4}{4} - \frac{4(0)^3}{3} + \frac{3(0)^2}{2}\right) -$$

$$\left[\frac{3^4}{4} - \frac{4(3)^3}{3} + \frac{3(3)^2}{2} - \left(\frac{1^4}{4} - \frac{4(1)^3}{3} + \frac{3(1)^2}{2}\right)\right]$$

$$= \frac{1}{4} - \frac{4}{3} + \frac{3}{2} \ - \left[\frac{81}{4} - \frac{108}{3} + \frac{27}{2} - \left(\frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right)\right]$$

$$= \frac{5}{12} + \frac{9}{4} + \frac{5}{12}$$

$$= \frac{37}{12}$$

Question 2

Evaluate

Marks 7

$$\int_0^\infty x \ e^{-x^2} \ dx$$

Solution:

$$\int_0^\infty x \ e^{-x^2} \ dx$$

$$= \lim_{t \to \infty} \int_0^t x \ e^{-x^2} \ dx$$

$$= -\frac{1}{2} \lim_{t\to\infty} \int_0^t (-2x)\ e^{-x^2}\ dx$$

$$= -\frac{1}{2} \lim_{t\to\infty} \left| e^{-x^2} \right|_0^t$$

$$= -\frac{1}{2}[0-1] = \frac{1}{2}$$

Question 3 **Marks 10**

Evaluate

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

Solution:

Consider,

$$\frac{1}{(x^2+a^2)(x^2+b^2)} = \frac{Ax+B}{x^2+a^2} + \frac{Cx+D}{x^2+b^2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\Rightarrow \quad 1 = (Ax+B)(x^2+b^2) + (Cx+D)(x^2+a^2)$$

$$\Rightarrow \quad 1 = Ax^3 + Bx^2 + Axb^2 + Bb^2 + Cx^3 + Dx^2 + Cxa^2 + Da^2$$

$$\Rightarrow \quad 1 = (A+C)x^3 + (B+D)x^2 + (Ab^2+Ca^2)x + (Bb^2+Da^2)$$

Comparing coefficients

$$A+C=0 \quad , \quad B+D=0$$

$$Ab^2+Ca^2=0 \quad , \quad Bb^2+Da^2=1$$

$$Here, \quad C=-A \Rightarrow Ab^2-Aa^2=0$$

$$\Rightarrow A(b^2-a^2)=0 \Rightarrow A=0 \quad if \quad b^2-a^2\neq 0$$

$$\Rightarrow C=0$$

$$Since, \quad B+D=0 \quad \Rightarrow \quad D=-B$$

$$thus \quad \Rightarrow \quad Bb^2-Ba^2=1 \quad \Rightarrow B(b^2-a^2)=1 \Rightarrow B=\frac{1}{(b^2-a^2)}$$

$$\Rightarrow D=\frac{-1}{(b^2-a^2)}$$

So (1) *becomes*

$$\frac{1}{(x^2+a^2)(x^2+b^2)}=\frac{(0)x+1/(b^2-a^2)}{x^2+a^2}+\frac{(0)x-1/(b^2-a^2)}{x^2+b^2}$$

$$=\frac{1}{(b^2-a^2)(x^2+a^2)}-\frac{1}{(b^2-a^2)(x^2+b^2)}$$

$$=\frac{1}{b^2-a^2}\left[\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right]$$

integrating both sides,

$$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}=\frac{1}{a^2-b^2}\left[\int_0^\infty \frac{dx}{(x^2+b^2)}-\int_0^\infty \frac{dx}{(x^2+a^2)}\right]$$

$$=\frac{1}{a^2-b^2}\lim_{t\to\infty}\left[\int_0^t \frac{dx}{(x^2+b^2)}-\int_0^t \frac{dx}{(x^2+a^2)}\right]$$

$$=\frac{1}{a^2-b^2}\lim_{t\to\infty}\left[\frac{1}{b}\tan^{-1}(\frac{x}{b})-\frac{1}{a}\tan^{-1}(\frac{x}{a})\right]_0^t$$

$$=\frac{1}{a^2-b^2}\lim_{t\to\infty}\left[\frac{1}{b}\tan^{-1}(\frac{t}{b})-\frac{1}{a}\tan^{-1}(\frac{t}{a})\right]$$

$$=\frac{1}{a^2-b^2}\lim_{t\to\infty}\left[\frac{\pi}{2b}-\frac{\pi}{2a}\right]=\frac{\pi}{2ab(a+b)}$$

Question No.1

Marks 10

Find the volume of the solid that results when the region enclosed by the given curve is revolved about the x-axis

$$y=\sqrt{25-x^2}, \quad y=3$$

Solution:

We know the volume formula, when we revolve $y = f(x)$ about x-axis, is given by

$$\int_a^b \pi([f(x)]^2 - [g(x)]^2)dx$$

Here given that $y = \sqrt{25 - x^2}$ *and* $y = 3$.

Now to find the initial and final limits of integrations we shall solve both of curves simultaneously to find their points of intersection. On equating both functions we have

$$\sqrt{25 - x^2} = 3$$
$$\Rightarrow \quad 25 - x^2 = 9$$
$$\Rightarrow \quad x^2 = 16$$
$$\Rightarrow \quad x = \pm 4$$
$$\Rightarrow \quad x = -4,\ 4$$

Hence given $a = -4,$ *and* $b = 4$, so we have

$$\int_a^b \pi([f(x)]^2 - [g(x)]^2)dx = \int_{-4}^4 \pi([\sqrt{25 - x^2}]^2 - [3]^2)dx$$
$$= 2\pi\int_0^4 \left(25 - x^2 - 9\right)dx = 2\pi\int_0^4 \left(16 - x^2\right)dx$$
$$= 2\pi\left[16x - \frac{x^3}{3}\right]\Bigg|_0^4$$
$$= 2\pi\left[64 - 64/3\right]$$
$$= \frac{256\pi}{3}$$

Question No. 2 Marks 8

Use cylindrical shells to find the volume of the solid generated when the region enclosed by the given curves is revolved about the y-axis.

$y=\sqrt{x},\ x=4, x=9, y=0$

Solution:

To find the volume when we revolve around y-axis, we integrate with respect to x i.e.

$$V=\int_a^b 2\pi x f\left(x\right)dx$$

the curve is $y=\sqrt{x}$ *so*

$$V=\int_a^b 2\pi x f\left(x\right)dx=\int_a^b 2\pi x[\sqrt{x}]dx$$

$Here\ \ a=4\ \ ,\ \ and\ \ \ b=9$

$Thus$

$$V=\int_4^9 2\pi x[\sqrt{x}]dx$$
$$=2\pi[\int_4^9 x^{3/2}dx$$
$$=844\pi/5$$

Question No.3

Find the area of the surface generated by revolving the given curve about the y –axis

$x=\sqrt{16-y},\ 0\le y\le 15$

Solution:

Here given that

$$x = g(y) = \sqrt{16 - y}, \qquad 0 \le y \le 15$$

$$\Rightarrow \quad g'(y) = -\frac{1}{2\sqrt{16 - y}}$$

$$\Rightarrow \quad (g'(y))^2 = \frac{1}{4(16 - y)}$$

$$\Rightarrow \quad 1 + (g'(y))^2 = 1 + \frac{1}{4(16 - y)}$$

$$\Rightarrow \quad = \frac{65 - 4y}{4(16 - y)}$$

$$\Rightarrow \quad \sqrt{1 + (g'(y))^2} = \sqrt{\frac{65 - 4y}{4(16 - y)}}$$

Since

$$S = \int_0^{15} 2\pi x \sqrt{1 + (g(y)')^2}\, dy$$

So, by above values

$$S = 2\pi \int_0^{15} \left(\sqrt{16 - y}\right)\left(\sqrt{\frac{65 - 4y}{4(16 - y)}}\right) dy$$

$$= (65\sqrt{65} - 5\sqrt{5})\frac{\pi}{6}$$

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